# THE EFFECTS OF LONGITUDINAL DEVIATORIC STRESS ON BOREHOLE TILTING AT BYRD STATION IN WESTERN ANTARCTICA

A Thesis

Presented in Partial Fulfillment of the Requirements
for the Degree Bachelors of Science

by

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## ABSTRACT

Borehole tilting measurements at Byrd Station in West Antarctica taken in 1975 have provided a data set against which mathematical ice modelling can be compared and adjusted for accuracy. The model described in this paper is an advance in rheological studies because it takes into consideration, the longitudinal stretching stress in addition to the shear stress acting on a horizontal plane. The data from Byrd Station and the predictions cast by the model are in very good agreement in terms of resultant ice velocity and the shear strain rate as a function of depth. Parameters in the flow law were the object of a sensitivity study, which revealed that the flow law constant A and the exponent n were the most important in adjusting the output of the model.

TABLE OF CONTENTS

LIST OF FIGURES:

LIST OF TABLES:

Page

LIST OF GRAPHS:

# I. INTRODUCTION

Much or all of the motion of the inland ice sheets is by internal shearing deformation. Laminar flow models were used by earlier workers (Nye 1959, Hooke 1981) to calculate the velocity-depth profile. Measurements at Byrd Station, however show what has been interpreted as unusual behavior and here we apply and test the laminar flow theory and also apply and test a theory incorporating longitudinal stretching.

According to the laminar flow model, layers or laminae move over one another in a shearing motion, much like a deck of cards pushed on a table top, with each card representing a layer. This model does not allow for deformation within the laminae themselves. The largest shearing action occurs at the base and the highest velocity is indicated at the ice sheet's surface. The laminar flow model is relatively simple and easy to apply but it includes only the shear stress acting on a horizontal plane and it increases linearly with depth (the other five deviatoric stresses are taken to be zero). For an example in which temperature is held constant with depth, the shear strain rate, or velocity gradient with respect to depth, increases as the nth power of depth, where n is the exponent in the flow law.

Measurements of borehole tilting at Byrd Station, West Antarctica (Garfield and Ueda, 1975) indicate much faster shearing at the shallow depths than the laminar flow theory predicts. A major limitation of the laminar flow model is the assumption that all other stresses, including the longitudinal stress are insignificant compared with the horizontal shear stress. Paterson (1983) and Lliboutry (1985) have looked at the borehole tilting at Byrd Station and have found discrepancies in trying to model the flow there. Measurements at the surface show significant longitudinal stretching, which implies that longitudinal stress may play an important role in the deformation of ice, perhaps considerably increasing the near surface shearing.

Calculating the longitudinal stress involves the solution of a system of equations. This is accomplished by using an inexpensive Fortran program on The Ohio State University's MVS computer system (see appendix 4).

This study will provide an improvement on the contemporary ice flow models, as well as a further test of 'Power Law Creep' theory. This theory is applied to the deformation of many plastic materials, of which rock and ice are two examples. The former, however, is not conducive to observational study due to the long periods of time required to have noticeable deformation. Ice on the other hand deforms readily under relatively low pressure and in short time, and thus provides a good analogy for verifying models on rock deformaton.

## II. THEORY

In our model, the horizontal velocity changes in both the horizontal, x, and the vertical, z, directions (figure 2-1); u represents the horizontal component of velocity (and x and z represent the horizontal and vertical coordinates, respectively). The vertical component of velocity is not required.

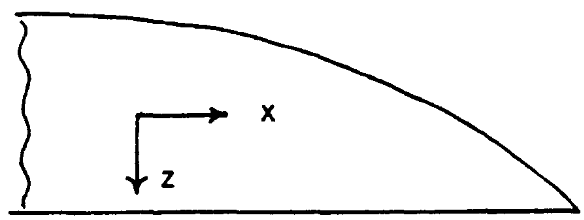



FIGURE 2-1. Vertical cross-section of a hypothetical glacier to illustrate the coordinate system used.

This study uses three central equations which describe the flow law. The horizontal and vertical velocity gradients can be expressed in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial z}$ respectively (Paterson 1975, p.31):

$$1) \qquad \frac{\partial u}{\partial x} = A\ \tau_e^{\,n-1}\ \sigma'_{xx}$$

$$2) \qquad \frac{\partial u}{\partial z} = 2A\ \tau_e^{\,n-1}\ \tau_{xz}$$

where n is a constant, generally accepted as being equal to 3, and A is a function of temperature (A and n are in themselves related and are discussed in appendix 2). Considering only the shear and longitudinal stretching stresses, $\tau_e$, the effective shear stress is defined by:

$$3) \qquad \tau_e = \tau_{xz}{}^2 + \sigma'_{xx}{}^2$$

which combines $\tau_{xz}$, the shear stress acting on a horizontal plane and $\sigma'_{xx}$, the longitudinal deviatoric stretching stress, due to the extensional flow of ice, and is taken to be parallel to the ice flow.

The shear stress acting on a horizontal plane, $\tau_{xz}$, can be expressed as:

$$4) \qquad \tau_{xz} = \rho g z \sin \alpha$$

where $\rho$ is the density of ice taken to be 920 kg/m$^3$, g is the acceleration due to gravity, z is the ice depth, and $\alpha$ is the surface slope of the ice sheet (see appendix 1). There is a more accurate form of this, but as C.J. Van der Veen (1985) has shown, the corrections are small.

The objectives here are to calculate the longitudinal stress, $\sigma'_{xx}$, and the horizontal velocity gradient with respect to depth $\frac{\partial u}{\partial z}$. The horizontal velocity gradient, $\frac{\partial u}{\partial x}$, is taken to vary with depth. According to arguments in Whillans (1979), we take:

$$5) \qquad \frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_o \left(1-\left(\frac{z}{Zmax}\right)^{p+1}\right)$$

where $(\frac{\partial u}{\partial x})_o$ is the measured surface stretching rate, z is the depth at which the velocity gradient is to be calculated, and Zmax is the total ice sheet thickness, in the case of Byrd Station, 2164 meters. P is an exponent that describes how longitudinal stretching varies with depth, and the equation causes it to be a maximum at the surface and zero at the bed. According to Whillans (1979), if the ice flow is planar, the exponent may be obtained from the horizontal velocity profile. Here however, we treat the value of P as arbitrary, and later discuss its sensitivity and results.

Equations 1, 3, 4, and 5 constitute 4 equations in 4 unknowns $(\frac{\partial u}{\partial x}, \tau_e, \tau_{xz}, \sigma'_{xx})$ which are solved for $\sigma'_{xx}$ and $\tau_e$ and used in equation 2 to obtain the shear that is to be compared with that measured in the borehole. The method of solution is described in appendix 3.

## III. RESULTS

The theory is used to slove for the various parameters in the flow law and one or more of the components can be changed in order to observe the sensitivity of the results. The following is a discussion of the results of the general model (with 'accepted' values for the adjustable parameters). Manipulations of some of the components of the general model are discussed in the sensitivity study.

The shear stress, $\tau_{xz}$, varies linearly (graph 1) with depth. The minimum value was realized at the surface because there is no stress acting at that interface due to a lack of friction. At the base of the glacier, there are equal and opposite forces at work, the shear stress is at its maximum around 3.9 E+4 Pascal, and is balanced by the frictional drag against the bed.

Graph 2 depicts the sinusoidal function of the longitudinal deviatoric stress curve. According to the model, the longitudinal stress plays a very significant role in the upper 1/3 of the glacier, tapering down to virtually nothing in the middle 1/3 of the profile. The upper portion appears to act like a skin on the ice sheet, very highly stressed. This is because the upper portion of the glacier is cold and stiffer, and the stretching is limited by the difficulty in deforming this upper layer.

The effective shear stress, $\tau_e$, is the square-root of the sum of the squares of $\tau_{xz}$ and $\sigma'_{xx}$. At the surface where $\tau_{xz}$ is zero, $\tau_e$ is equal to $\sigma'_{xx}$. At the base the stretching stress, $\sigma'_{xx}$, is equal to zero and so $\tau_e$ is equal to $\tau_{xz}$. The wavering in the middle is due to the 'S' shape of the $\sigma'_{xx}$ function.

The velocities are referenced with respect to the surface of the ice sheet (looking down the borehole) to correspond with the data from Garfield and Ueda (1975). Comparison with the measurements is discussed in section V.

The horizontal strain rate (graph 5) is a function as defined by Whillans (1979) and varies in a simple curvilinear fashion, decreasing from its maximum of 1.69 E-11/sec at the surface to zero at the bed.

The value of $\frac{\partial u}{\partial z}$, the velocity gradient with respect to depth, on the other hand goes from nearly zero at the surface to its maximum of 5.9 E-10/sec at the base with a shape unlike that of the horizontal strain rate. In the lower 1/3 of the profile it levels out from a rapid decline with depth as it reaches the base.

## IV. SENSITIVITY OF CALCULATIONS

Considered in the sensitivity study are the surface slope, $\alpha$, the temperature dependent flow law constant, A, and the exponent, n, for the flow law. In addition the equation chosen to represent the variation of horizontal strain rate with depth is compared with another possible equation to observe the difference.

The original values for $\alpha$ and n are lessened and increased to check their effects on their resultant stresses, strain rates, and velocities. For the purpose of Table 1, it is necessary to choose some of the functions to gauge the variation; the effective shear stress and the shear strain rate are chosen because they incorporate both the shear stress acting on a horizontal plane and the longitudinal deviatoric stress by definition.

The new and old values are compared for a percent deviation from the "acceptable" values, as follows:

$$\frac{(\text{Accepted value}) - (\text{New value})}{(\text{Accepted value})} \times 100 = \%\ \text{Change}$$

TABLE 1: Sensitivity of Parameters on Effective Stress

This table shows the effects of changing one of the parameters in the model. The percent change is measured with respect to the calculated values for the effective shear stress, because this function incorporates both the shear stress acting on a horizontal plane and the longitudinal deviatoric stress.

| PARAMETER | ORIGINAL VALUE | NEW VALUE | % CHANGE AT DEPTH (Effective Stress) 100 m | 1000 m |
|---|---|---|---|---|
| Surface Slope | 0.002 radians | 0.001 | 0.0% | +1.1% |
| | 0.002 radians | 0.003 | 0.0% | -1.9% |
| Flow Law Exponent | n=3 | n=1 | +60.4% | +53.6% |
| | n=3 | n=2 | +20.7% | +20.5% |
| | n=3 | n=4 | -12.2% | -12.7% |
| Strain Eq. Exponent | P=2 | P=1 | +0.2% | +4.2% |
| | P=2 | P=3 | 0.0% | -1.9% |
| Surface Stretching | $(\frac{\partial u}{\partial x})_o =$ | $(\frac{\partial u}{\partial x})_o = 0$ Laminar Flow | +97.17% | +70.7% |

TABLE 2: Sensitivity of Parameters on Strain Rate

This table shows the effects'of changing one of the parameters in the model on the strain rate or velocity gradient with respect to depth.

| PARAMETER | ORIGINAL VALUE | NEW VALUE | % CHANGE AT DEPTH (Shear Strain Rate) 100 m | 1000 m |
|---|---|---|---|---|
| Surface Slope | 0.002 radians | 0.001 | +50.0% | +51.1% |
| | 0.002 radians | 0.003 | -50.4% | -51.9% |
| Flow Law Exponent | n=3 | (n=1 | -153.1% | -165.6%) |
| | n=3 | n=2 | -26.6% | -29.0% |
| | n=3 | n=4 | -11.0% | -12.0% |
| Strain Eq. Exponent | P=2 | P=1 | +0.1% | +8.4% |
| | P=2 | P=3 | 0.0% | -2.3% |
| Surface Stretching | $(\frac{\partial u}{\partial x})_o = 1.7E-11$ | $(\frac{\partial u}{\partial x})_o = 0$ Laminar Flow | +99.2% | +91.4% |

## V. MEASUREMENTS OF BOREHOLE TILTING

Tilting data from the Byrd Station borehole are described by Garfield and Ueda (1975), in terms of displacement during a seven year period. In addition to horizontal displacement, the azimuth of movement was measured using a Parsons multishot inclinometer. Two important points should be noted: the drill was lost, blocking the borehole at a depth of 1474 meters, and so the data cover only 68% of the profile. Secondly, casual inspection of the data suggest that the ice flow is in a cork-screw fashion rather than planar flow as assumed in most 2-dimensional models.

Using these data, in conjunction with the difference between the azimuth of movement and the direction of ice flow, it is possible to isolate the x and y components of velocity with the equations:

$$Ux = (\sin \Theta)\ (\Delta)/T$$

$$Uy = (\cos \Theta)\ (\Delta)/T$$

where Ux and Uy represent the x and y components of velocity, $\Theta$ is the difference between the azimuth of movement and the flow line, and $\Delta$ is the total horizontal displacement. The time between measurements, T, is 7.0 years.

After calculation of the components of velocity, it is possible to calculate the strain rate with respect to depth for each of the components:

$$\frac{\partial Ux}{\partial z} = (Ux0-Ux1)/(Z0-Z1)$$

$$\frac{\partial Uy}{\partial z} = (Uy0-Uy1)/(Z0-Z1)$$

where $\frac{\partial Ux}{\partial z}$ represents the gradient with respect to depth of the x-component of velocity, Ux0 is the displacement at Z0 and Ux1 is the displacement at Z1. And similarily in the second equation $\frac{\partial Uy}{\partial z}$ represents the gradient with respect to depth of the y-component of velocity.

The two gradients $\frac{\partial Ux}{\partial z}$ and $\frac{\partial Uy}{\partial z}$ are shown in graphs 8 and 9. The x-component plot depicted in graph 8 seems to have a better point distribution, and its basic shape follows that predicted by the model down to the depth at which the Byrd Station borehole is blocked. The data graph and the corresponding graph for the model fit well, with less than a factor of five discrepancy in comparing the strain rates (graphs 8 and 10, respectively).

## VI. DISCUSSION

The measured borehole tilting from Byrd Station seems to agree with the model. The velocity as calculated in each case shows agreement within a factor of two, at the base of the Byrd Station data (1474 meters). The standard model predicts velocities which are higher than those measured.

The values of the vertical strain rate with respect to depth in the Byrd Station data are scattered due to the suggested cork-screw fashion flow of the ice there. For this reason the x and y components of the ice velocity are separated, but still the points are scattered. The upper portion of the x-component curve fits well with the upper portion of the predicted curve.

Through the manipulation of the parameters in the flow law, the model can allow for the decrease in velocity to match the data from Byrd Station. The n=3 standard can be changed to n=1 to achieve a velocity of correct magnitude.

## VI. CONCLUSIONS

The flow law, taking deviatoric stress into consideration, gives a reasonable approximation of the ice flow due to internal deformation, and provides an excellent approximation of the vertical shear strain rate. The parameters are easily adjusted to fit the output with field measurements. The surface slope and the exponent in the horizontal strain rate equation have only a minor effect on the deviatoric stress, effective stress, and the resulting velocity profile. The values for the exponent, n, and constant, A, in the flow law appear to be most crucial in determining the deformation and predicting velocity and borehole tilting. The A(n) relationship, that is the relationship of the flow law constant A, as a function of the exponent n, is the most difficult to work with because the value for A is function of temperature, crystal size and preferential orientation, and other variables.

In developing the model, we are able to allow for the variation of horizontal strain with respect to depth, and temperature (including the calculation of A(n) for any given depth) and simultaneously solve for the shear stress acting on a horizontal plane and the longitudinal stretching stress. The model provides realistic values for the ice velocity and the vertical strain rate with respect to depth as is illustrated in the sensitivity study (section IV).

Difficulties in the model are encountered in that the borehole data are not complete to the base of the glacier, and defining a value for A(n) is difficult due to the existance of different types of ice in the Byrd Station core. The borehole data is of limited value due to the blockage of the hole at a depth of 1474 meters. This is a major stumbling block in trying to fit a model to the data because the most significant changes in the velocity, stress distribution, and stain rates probably occur in the lower 30% of the ice mass. Also, in trying to model a non-ideal system such as Byrd Station, four different types of ice were identified in the core: Holocene ice with air bubbles, bubble-free Holocene ice, fine grained Wisconsinan ice with a strong single maximum fabric, and a large grained Wisconsinan ice with a multiple maximum fabric (Paterson, 1983).

The successful application of this model in describing ice flow at Byrd Station gives it credibility. The continued study of this topic is in order, possibly adapting the model to the data from Camp Century, Greenland, where a complete data set is available. This would be beneficial in describing the flow in the lower portion of the ice mass.

## APPENDIX 1: Derivation of Shear Stress Equation

It is the purpose of this appendix to justify the generally accepted equation for stress.

$$\tau_{xz} = \rho g z \ (\sin \alpha)$$

The shear stress, $\tau_{xz}$, acting on a horizontal plane xy, is a linear function of depth z where $\rho$ is the density of ice, g is the acceleration due to gravity, and $\alpha$ is the surface slope of the ice sheet.

In this system there are three forces acting on the glacier (figure A1.1). The first is the normal force, N, due to gravity acting perpendicular to the slope, and the second set is the shear force working perpendicular to the normal force and approximately parallel to flow.

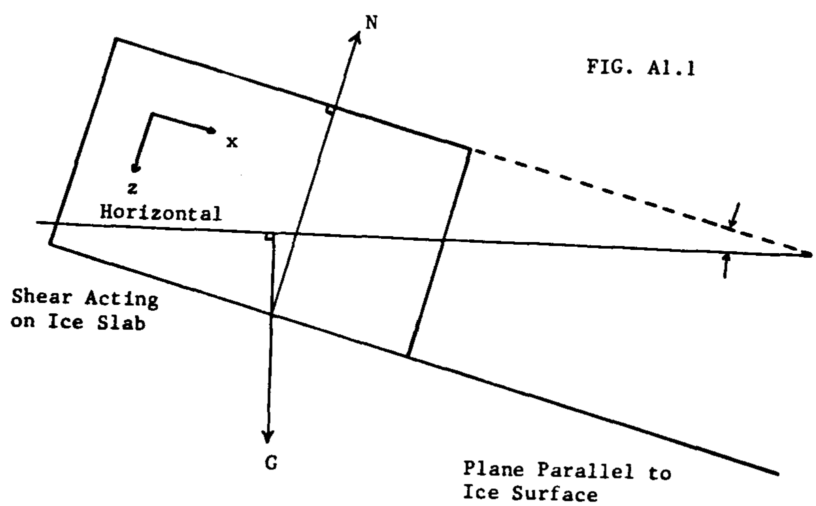



FIG. A1.1

The third force is due to gravity, G:

1) $G = mg$

where m is the mass of the ice. Because there is no acceleration, these three forces must sum to zero. The x-component of gravitational force is therefore balanced by T and the z-component by N.

2) $T = mg\ (\sin \alpha)$

3) $N = mg\ (\cos \alpha)$

T is the shear force and because shear stress is defined as shear force per unit area it is possible to substitute density and ice thickness in place of mass in equation 2.

density = mass/volume

Volume, V, is the product of the x, y, and z lengths, $\Delta x$, $\Delta y$, and z:

4) $V = \Delta x\ \Delta y\ z$

and multiplying by density, $\rho$, provides the mass:

5) $m = \rho\ \Delta x\ \Delta y\ z$

Substituting this expression of mass into equation 2:

6) $T = \rho\ \Delta x\ \Delta y\ z\ g\ (\sin \alpha)$

Since the shear stress, $\tau_{xz}$, is the shear force per unit area, in this case $\Delta x \; \Delta y$, the area term cancels and we have:

$$7) \qquad \tau_{xz} = \rho g z \; (\sin \alpha)$$

which derives in essence, from the balance of the x-component of gravity with traction. Differences in longitudinal pushes and pulls along the slab are neglected.

The depth, z, is arbitrary and so the shear stress varies with depth as indicated in equation 7. That is the thicker the slab, the heavier it is and the larger must be the traction of shear stress to support it.

## APPENDIX 2: The Flow Law Constant

The function A in the flow law is a constant describing the viscosity of stiffness of ice and its resistance to flow. A depends on a number of factors which include ice temperature, crystal size and preferential orientation in the ice, impurities (such as dust) and perhaps other parameters (Paterson, 1975). According to the Arrhenius relation A is defined:

$$A = A_o \exp(-Q/RT)$$

where $A_o$ is an initial value for A, independent of temperature, Q is the activation energy for creep of ice, R is the natural gas constant, and T is the temperature in Kelvin. The activation energy for creep of ice is temperature dependent. A value of 60000 J/mol is used for ice below -10°C (Weertman, 1973) and the natural gas constant is 8.314 J/mol-K.

A value for $A_o$ is calculated from Paterson's table of values for A(3) as a function of temperature, where n=3 (page 39). In order to keep n as a variable in this model, it is necessary to understand the relationship between A(n) and n, where A(n) is a value calculated for A for a given n.

In simplified form the flow law is expressed:

$$\dot{\varepsilon}_{(n)} = A_{(n)}\, \tau_e^{\,n-1}\, \sigma'_{xx}$$

where $\dot{\varepsilon}_{(n)}$ is the strain rate, $\tau_e$ is the effective shear stress, and $\sigma'_{xx}$ is the deviatoric stress.

In the case of n=3, Paterson's equation would be:

$$\dot{\varepsilon}_{(3)} = A_{(3)}\,\tau_e^{\,2}\quad\sigma'_{xx}$$

For the special case $\dot{\varepsilon}_{(3)} = \dot{\varepsilon}_{(n)}$, where $\tau_e = \sigma'_{xx}$ = 'ONE', some arbitrary value, we can derive the relationship of $A_{(3)}$ to $A_{(n)}$. Setting the equations equal:

$$A_{(3)}\,\tau_e^{\,2}\quad\sigma'_{xx} = A_{(n)}\,\tau_e^{\,n-1}\,\sigma'_{xx}$$

$$A_{(3)}\,(\mathrm{ONE})^2\,(\mathrm{ONE}) = A_{(n)}\,(\mathrm{ONE})^{n-1}\quad(\mathrm{ONE})$$

Isolating $A_{(n)}$:

$$A_{(n)} = A_{(3)}\quad\frac{(\mathrm{ONE})^2}{(\mathrm{ONE})^{n-1}}$$

Simplifying we get:

$$A_{(n)} = A_{(3)}\,(\mathrm{ONE})^{3-n}$$

When $A_{(n)}$ is used in the flow law, the complex units cancel and leave the resultant strain rate per unit time.

APPENDIX 3: Solving for Longitudinal Stress

In the process of solving for the velocity gradient with respect to depth $\frac{\partial u}{\partial z}$, we must know the effective shear stress, $\tau_e$, for which we need to know the longitudinal stress, $\sigma'_{xx}$. The depth variation in the horizontal velocity gradient or strain rate, $\frac{\partial u}{\partial x}$, is presc The strain rate is taken to change with depth in a non-linear fashion:

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right)_o \left(1 - \frac{z}{Zmax}\right)^{P+1}$$

where $\left(\frac{\partial u}{\partial x}\right)_o$ is the measured surface strain rate, z is the depth in the ice, Zmax is the total ice thickness, and P is an exponent, which is examined in the sensitivity study. Whillans (1979) favors P=1.

We can now solve for the values of $\sigma'_{xx}$ at any depth according to equation:

$$\frac{\partial u}{\partial x} = A \, \tau_e^{\,n-1} \, \sigma'_{xx}$$

Substituting:

$$\tau_e = \sqrt{\tau_{xz}^{\,2} + \sigma'^{\,2}_{xx}}$$

this is rewritten as:

$$\frac{\partial u}{\partial x} = A \left(\sqrt{\tau_{xz}^{\,2} + \sigma'^{\,2}_{xx}}\right)^{n-1} \sigma'_{xx}$$

For n=3 this is a cubic equation in $\sigma'_{xx}$ with only one real root. Kosteka (1985) solved this equation in his calculation of stress distribution in southern Greenland. For several values of n, we use an iterative procedure starting with an estimate, $\sigma_{est}$, for $\sigma'_{xx}$. The equation is solved for the first order letting correction, $\Delta = \sigma'_{xx} - \sigma_{est}$. As $\sigma_{est}$ draws closer to $\sigma'_{xx}$, $\Delta$ will approach zero.

Substituting $\sigma'_{xx} = \sigma_{est} + \Delta$:

$$\frac{\partial u}{\partial x} = A \left(\sqrt{\tau_{xz}{}^2 + (\sigma_{est} + \Delta)^2}\right)^{n-1} (\sigma_{est} + \Delta)$$

which expands to:

$$\frac{\partial u}{\partial x} A^{-1} = \left(\tau_{xz}{}^2 + \sigma_{est}{}^2 + 2\,\sigma_{est}\,\Delta + \Delta^2\right)^{\frac{n-1}{2}} (\sigma_{est} + \Delta)$$

Since $\Delta$ will ultimately be very small, $\Delta^2$ is negligible so we can omit it by considering $\Delta^2 = 0$:

$$\frac{\partial u}{\partial x} A^{-1} = \left(\tau_{xz}{}^2 + \sigma_{est}{}^2 + 2\,\sigma_{est}\,\Delta\right)^{\frac{n-1}{2}} (\sigma_{est} + \Delta)$$

Because we wish to include different values for n in the sensitivity study, it must remain a variable, so by using the binomial expansion (CRC Handbook, 1979):

$$(x+y)^m = x^m + mx^{m-1}\,y + m\,\frac{(m-1)}{2}\,x^{m-2}\,y^2 . . . .$$

let
$$x = (\tau_{xz}^{\ 2} + \sigma_{est}^{\ 2})$$
$$y = \sigma_{est}\ \Delta$$
$$m = \frac{n-1}{2}$$

Substituting we get:

$$\frac{\partial u}{\partial x} A^{-1} = \left[(\tau_{xz}^{\ 2} + \sigma_{est}^{\ 2})^{\frac{n-1}{2}} + \frac{(n-1)}{2} (\tau_{xz}^{\ 2} + \sigma_{est}^{\ 2})^{\frac{n-1}{2} - 1} (2\ \sigma_{est}\ \Delta)\right.$$

$$\left. + \frac{\frac{n-1}{2}\left(\frac{n-1}{2} - 1\right)}{2} (\tau_{xz}^{\ 2} + \sigma_{est}^{\ 2})^{\frac{n-1}{2} - 2} (2\ \sigma_{est}\ \Delta)^2\right] (\sigma_{est} + \Delta)$$

and neglecting terms $\Delta^2$ and higher order

$$\frac{\partial u}{\partial x} A^{-1} = \left[(\tau_{xz}^{\ 2} + \sigma_{est}^{\ 2})^{\frac{n-1}{2}} + (n-1)\ (\tau_{xz}^{\ 2} + \sigma_{est}^{\ 2})^{\frac{n-3}{2}} (\sigma_{est}\ \Delta\right]$$

$$(\sigma_{est} + \Delta)$$

Completing the multiplication:

$$\frac{\partial u}{\partial x} A^{-1} = \left[\sigma_{est}\ (\tau_{xz}^{\ 2} + \sigma_{est}^{\ 2})^{\frac{n-1}{2}}\right] + \left[(n-1)\ (\tau_{xz}^{\ 2} + \sigma_{est}^{\ 2})^{\frac{n-3}{2}}\right.$$

$$\left.\sigma_{est}^{\ 2}\ \Delta\right] + \left[\Delta\ (\tau_{xz}^{\ 2} + \sigma_{est}^{\ 2})^{\frac{n-1}{2}}\right] + \left[(n-1)\right.$$

$$\left.(\tau_{xz}^{\ 2} + \sigma_{est}^{\ 2})^{\frac{n-3}{2}}\ \sigma_{est}\ \Delta^2\right]$$

Grouping the second and third terms and neglecting terms with $\Delta^2$:

$$\frac{\partial u}{\partial x} A^{-1} = \sigma_{est} \left[ \left(\tau_{xz}^2 + \sigma_{est}^2\right)^{\frac{n-1}{2}} + \Delta \left[ (n-1) \left(\tau_{xz}^2 + \sigma_{est}^2\right)^{\frac{n-3}{2}} \right. \right.$$

$$\left. \sigma_{est}^2 + \left(\tau_{xz}^2 + \sigma_{est}^2\right)^{\frac{n-1}{2}} \right]$$

Moving the first term to the lefthand side:

$$\frac{\partial u}{\partial x} A^{-1} - \sigma_{est} \left(\tau_{xz}^2 + \sigma_{est}^2\right)^{\frac{n-1}{2}} = \Delta \left[ (n^{-1}) \left(\tau_{xz}^2 + \sigma_{est}^2\right)^{\frac{n-3}{2}} \right.$$

$$\left. \sigma_{est}^2 + \left(\tau_{xz}^2 + \sigma_{est}^2\right)^{\frac{n-1}{2}} \right]$$

Isolating $\Delta$ on the lefthand side of the equation we get:

$$\Delta = \frac{\frac{\partial u}{\partial x} A^{-1} - \sigma_{est} \left(\tau_{xz}^2 + \sigma_{est}^2\right)^{\frac{n-1}{2}}}{(n^{-1}) \left(\tau_{xz}^2 + \sigma_{est}^2\right)^{\frac{n-3}{2}} \sigma_{est}^2 + \left(\tau_{xz}^2 + \sigma_{est}^2\right)^{\frac{n-1}{2}}}$$

If $\Delta$ is more than 2% of $\sigma'_{xx}$ then $\sigma_{est}$ is adjusted and a new calculation is made.

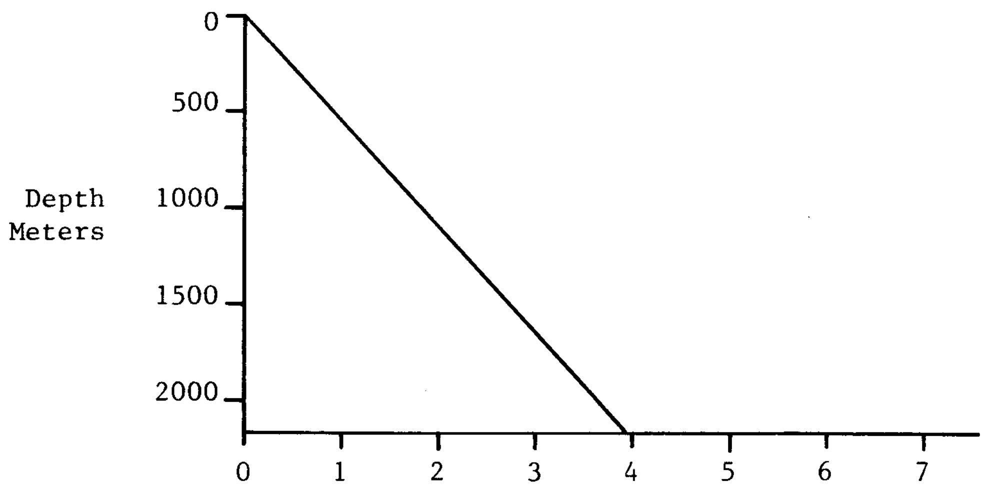
APPENDIX 4: Computer Model Code

```
//   JOB   CLASS=V
$JOB
C
C   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
C   %                                                                %
C   %               JEFFREY W.   DE FREEST                           %
C   %               INSTITUTE OF POLAR STUDIES                       %
C   %               125\SOUTH OVAL MALL                              %
C   %               COLUMBUS, OHIO    43210                          %
C   %                                                                %
C   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
C
C
C
C   ##################################################################
C   #                                                                #
C   #               GGGGG    L        A     CCCCC        1           #
C   #               G        L      A   A   C           11           #
C   #               G  GG    L      AAAAA   C            1           #
C   #               GGGGG    LLLL A     A   CCCCC      11111         #
C   #                                                                #
C   ##################################################################
C
C
C           THIS PROGRAM IS DESIGNED TO CALCULATE TO LONGITUDINAL
C           STRESS IN AN ICE SHEET, AND THEN GO ON TO SOLVE FOR
C           THE HORIZONTAL VELOCITY GRADIENT (OR STRAIN RATE) WITH
C           RESPECT TO DEPTH. MANIPULATEABLE PARAMETERS IN THIS PRO-
C           GRAM INCLUDE:
C                   1) THE CONSTANT 'A',AND ITS DERIVATION
C                   2) THE EXPONENT 'N',AS USED IN THE FLOW LAW
C
C
C
    0 REAL N
    5 WRITE(6,200)
   10 READ(5,190)Z,TEMP
   15 IF(Z.EQ.9999)STOP
   20 ALPHA=0.002
   25 GRAVTY=9.8
C                   GRAVITY IN METERS PER SECOND SQUARED
   30 RHO=920.
C                   DENSITY IN KILOGRAMS PER METER CUBED
   35 N=3.
   36 ONE=100000.
   40 P=2.
   45 THICK=2164.
C                   DEPTH IN METERS
   50 DUDXO=1.68E-11
   55 DUDZL=0.
   60 U=0.
   65 ZL=0.
   70 TAUXZ=RHO*GRAVTY*Z*SIN(ALPHA)
C                   SHEAR STRESS IN PASCALS
   75 DUDX=DUDXO*(1.-((Z/THICK)**(P+1.)))
C                   STRAIN RATE PER SECOND
   80 SIGXP=10000.
C                   DEVIATORIC STRESS IN PASCALS
   85 R=8.314
```

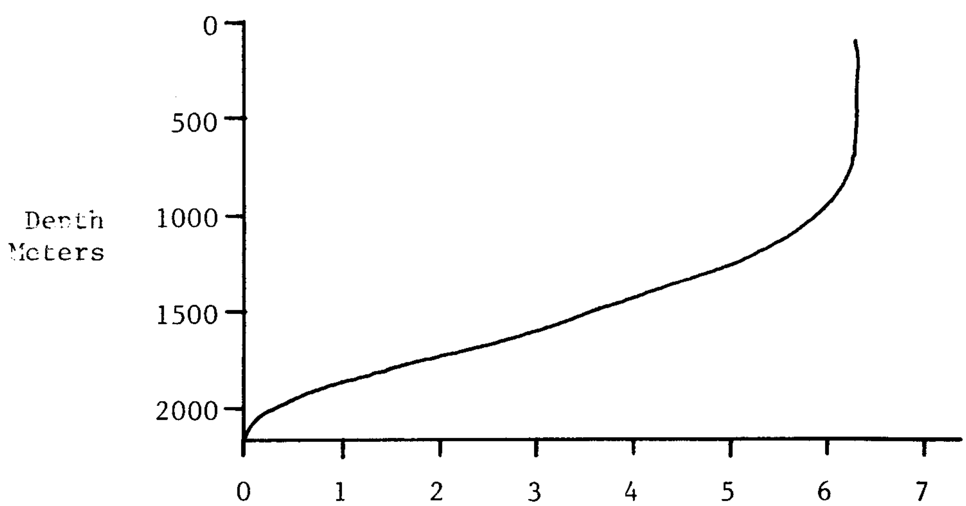
```
C                    NATURAL GAS CONSTANT IN JOULES PER MOLE-KELVIN
   90 Q=60000.
C                    ACTIVATION ENERGY FOR CREEP IN JOULES PER MOLE
   95 ABEGIN=4.4E-13
  100 A3=ABEGIN*(EXP(-Q/(R*(TEMP+273.))))
C
C           THE VALUE OF A FOR N=3,A3, IS CHANGED TO ACCOMODATE DIFF
C           IFFERET VALUES FOR N SO THAT STRAINRATE IS THE SAME AT A
C           STRESS EQUAL TO "ONE".
C
  104 A=A3*ONE**(3.-N)
  105 EQONE=DUDX/A-(SIGXP*(TAUXZ**2.+SIGXP**2.)**((N-1.)/2.))
  110 EQTWO=(N-1.)*(TAUXZ**2.+SIGXP**2.)**((N-3.)/2.)*SIGXP**2
  111 EQTHRE=(TAUXZ**2.+SIGXP**2.)**((N-1.)/2.)
  115 DELTA=EQONE/(EQTWO+EQTHRE)
  122 SIGXP=SIGXP+DELTA
  124 IF(ABS(SIGXP).LE.0.01E-25)GOTO135
  125 IF(ABS(DELTA/SIGXP).LE.0.02)GOTO135
  130 GOTO105
  135 TAUEFF=SQRT(TAUXZ**2.+SIGXP**2.)
  145 DUDZ=2*A*(TAUEFF**(N-1.))*TAUXZ
  150 U=U+((DUDZ+DUDZL)*(Z-ZL))/2
  160 UPYEAR=U*3.156E7
  170 WRITE(6,205)Z,TEMP,A,TAUXZ,SIGXP,TAUEFF,DUDX,DUDZ,UPYEAR
  175 ZL=Z
  180 DUDZL=DUDZ
  185 GOTO10
  190 FORMAT(F5.0,F6.2)
  200 FORMAT('1',/,3X,'DEPTH',4X,'TEMP',8X,'A',10X,'TAUXZ',8X,'SIGXP',8X
     >,'TAUEFF',8X,'DUDX',6X,'DUDZ',7X,'VELOCITY')
  205 FORMAT(3X,F5.0,3X,F6.2,3X,E10.3,3X,E10.3,3X,E10.3,3X,E10.3,3X,E10.
     >3,3X,E10.3,3X,E10.3)
      END
$ENTRY
  100-28.4
  200-28.4
  300-28.4
  400-28.5
  500-28.7
  600-28.8
  700-28.9
  800-28.8
  900-28.7
 1000-28.3
 1100-27.6
 1200-26.5
 1300-25.3
 1400-23.5
 1500-21.5
 1600-19.3
 1700-16.8
 1800-13.3
 1900-10.0
 2000-06.0
 2100-03.0
 2164-01.9
 9999-00.0
*/
//
```

```
//  JOB   CLASS=V
$JOB
C
C
C
C       BBBBBB   OOOOO   RRRRR   EEEEE   H     H   OOOOO   L       EEEEE
C        B    B  O    O  R    R  E       H     H   O    O  L       E
C        BBBBB   O    O  RRRRR   EEEEE   HHHHHH   O    O  L       EEEEE
C        B    B  O    O  R   R   E       H     H   O    O  L       E
C       BBBBBB   OOOOO   R    R  EEEEE   H     H   OOOOO   LLLLL   EEEEE
C
C  ************************************************************************
C
C                JEFFREY W. DEFREEST
C                INSTITUTE OF POLAR STUDIES
C                125 SOUTH OVAL MALL
C                COLUMBUS, OHIO        43210
C
C
C  ************************************************************************
C
C
C
C          THIS PROGRAM CONVERTS THE BOREHOLE TILTING DATA COLLECTED
C          BY GARFIELD AND UEDA  AT BYRD STATION ANTARCTICA(1975) INTO
C          MKS FOR APPLICATION IN A COMPARATIVE STUDY TO THEORETICAL
C          DATA GENERATED IN THE GLAC1 AND GLAC3 PROGRAMS
C
C
C
      DEPTH1=0.
      VELOE1=0.
      VELON1=0.
      VELOT1=0.
C  CHECK THE REFERENCE AZIMUTH BELOW:
C REFAZ IS THE DIFFERENCE BETWEEN TRUE NORTH AND FLOW DIRECTION
      REFAZ =26.0
   10 WRITE(6,200)
   20 READ(5,400)IFEET,DELXFT,AZIM
      ZFEET=FLOAT(IFEET)
   30 IF(IFEET.EQ.9999)STOP
      AZIRAD=(AZIM-REFAZ)*3.14159/180.
   40 DEPTH=ZFEET*0.3048
   50 DELXM=DELXFT*0.3048
   60 TIME=7.
   80 VELOE=DELXM/TIME*COS(AZIRAD)
  100 DUDZ=(VELOE-VELOE1)/(DEPTH-DEPTH1)
      VELON=DELXM/TIME*SIN(AZIRAD)
      DVDZ=(VELON-VELON1)/(DEPTH-DEPTH1)
      VELOT=DELXM/TIME
      DTDZ=(VELOT-VELOT1)/(DEPTH-DEPTH1)
      DELAZ=ATAN2(VELOE,VELON)*180./3.14159
  110 WRITE(6,300)ZFEET, DELXFT, AZIM, DEPTH,VELOE,DUDZ,VELON,DVDZ,VELOT
     >,DTDZ,DELAZ
  120 VELOE1=VELOE
      VELON1=VELON
      VELOT1=VELOT
  130 DEPTH1=DEPTH
  140 GOTO20
```

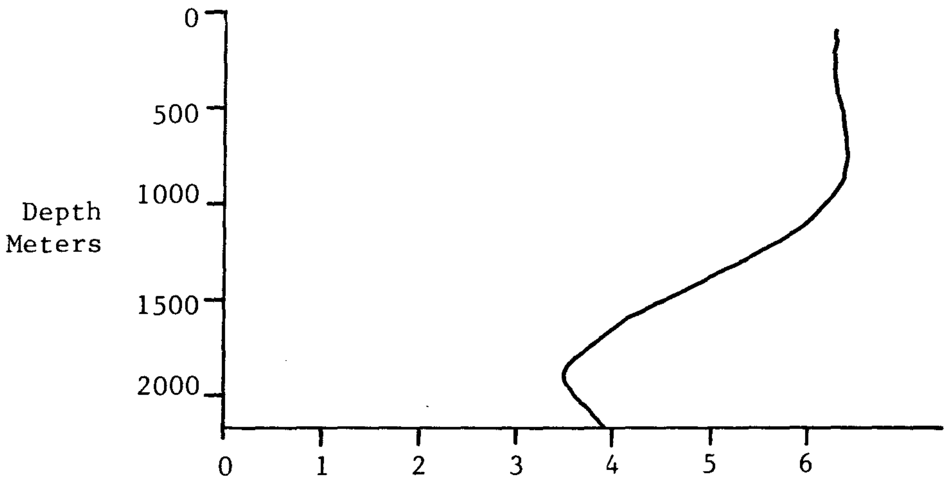
```
  200 FORMAT(2X,'FEET',3X,'XFT',2X,'AZIM',2X,'DEPTH',2X,'VELOE',4X,'DUDZ
     >',4X,'VELON',4X,'DVDZ',5X,'VELCT',3X,'DTDZ')
  300 FORMAT(1X,3F6.1,F7.2,3(F7.3,E10.3),F6.1)
  400 FORMAT(I5,F5.1,F3.0)
      END
$ENTRY
  300  1.3201
  400  1.8201
  497  1.3201
  606  0.7223
  706  1.3230
  895  0.5259
  994  0.9036
 1100  1.5038
 1198  1.8046
 1294  2.5063
 1392  3.3088
 1499  4.0106
 1591  4.6113
 1641  5.2116
 1690  6.0118
 1740  6.9121
 1790  7.6122
 1840  8.2125
 1890  8.4125
 1939  8.9129
 1989  9.5134
 2039 10.4138
 2088 11.3139
 2138 12.5139
 2188 13.2138
 2287 15.2140
 2386 16.4133
 2484 17.5131
 2582 18.7129
 2682 19.9124
 2780 21.1119
 2878 22.4113
 2976 24.0108
 3074 25.5104
 3172 26.8100
 3270 27.8095
 3368 28.6091
 3446 29.4084
 3564 30.7076
 3662 31.7074
 3759 31.5069
 3857 31.3063
 3955 31.2060
 4053 31.9056
 4150 33.7051
 4249 35.6045
 4346 38.3041
 4444 40.3040
 4542 41.2040
 4639 43.4038
 4738 47.1034
 4835 51.2030
 9999
999
```



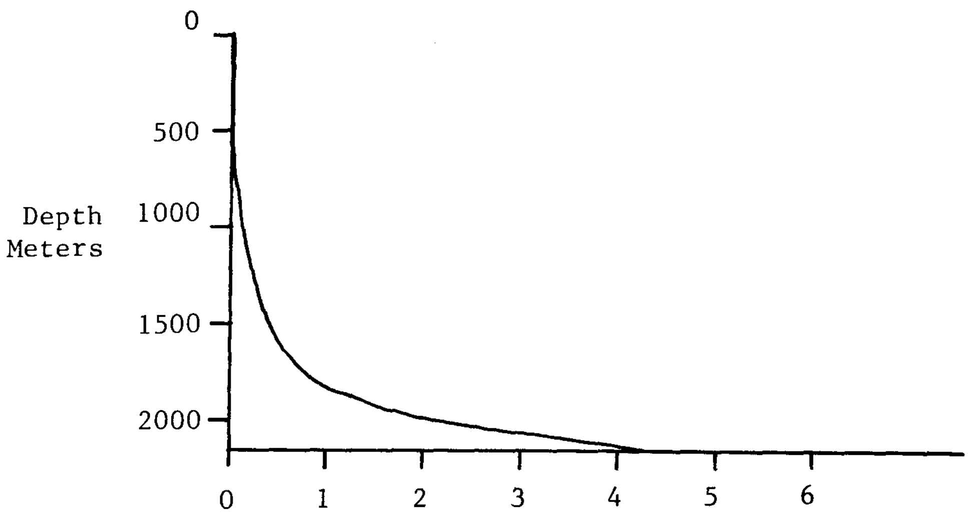

GRAPH 1: Shear Stress Pascals X 1000



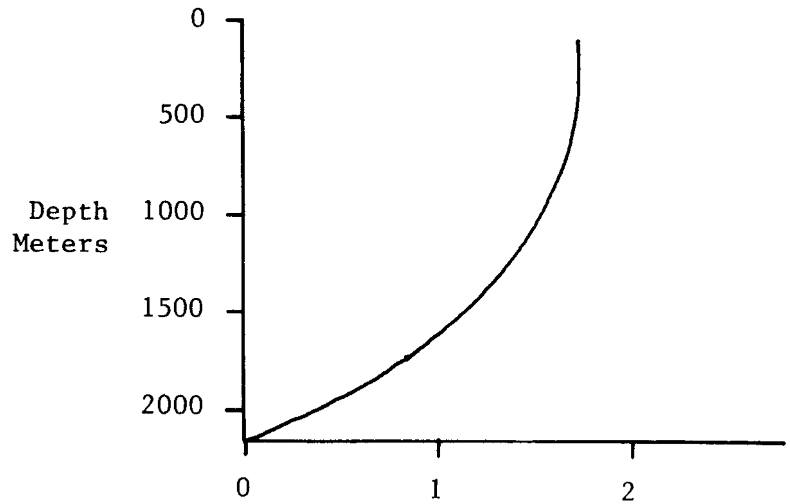

GRAPH 2: Deviatoric Stress Pascals X 1000



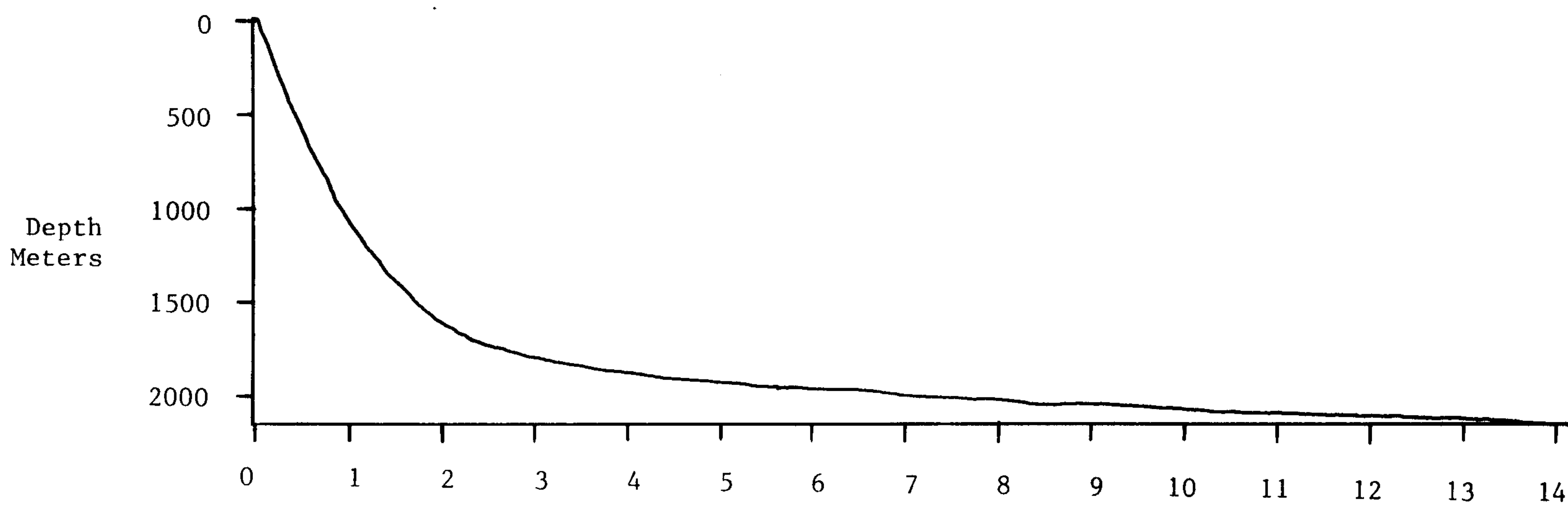

GRAPH 3: Effective Stress Pascals X 1000



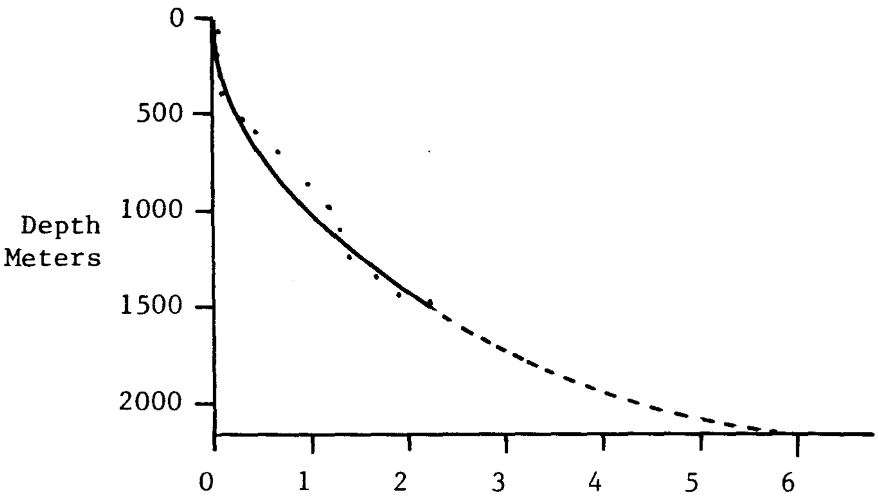

GRAPH 4: Velocity Meters/Year



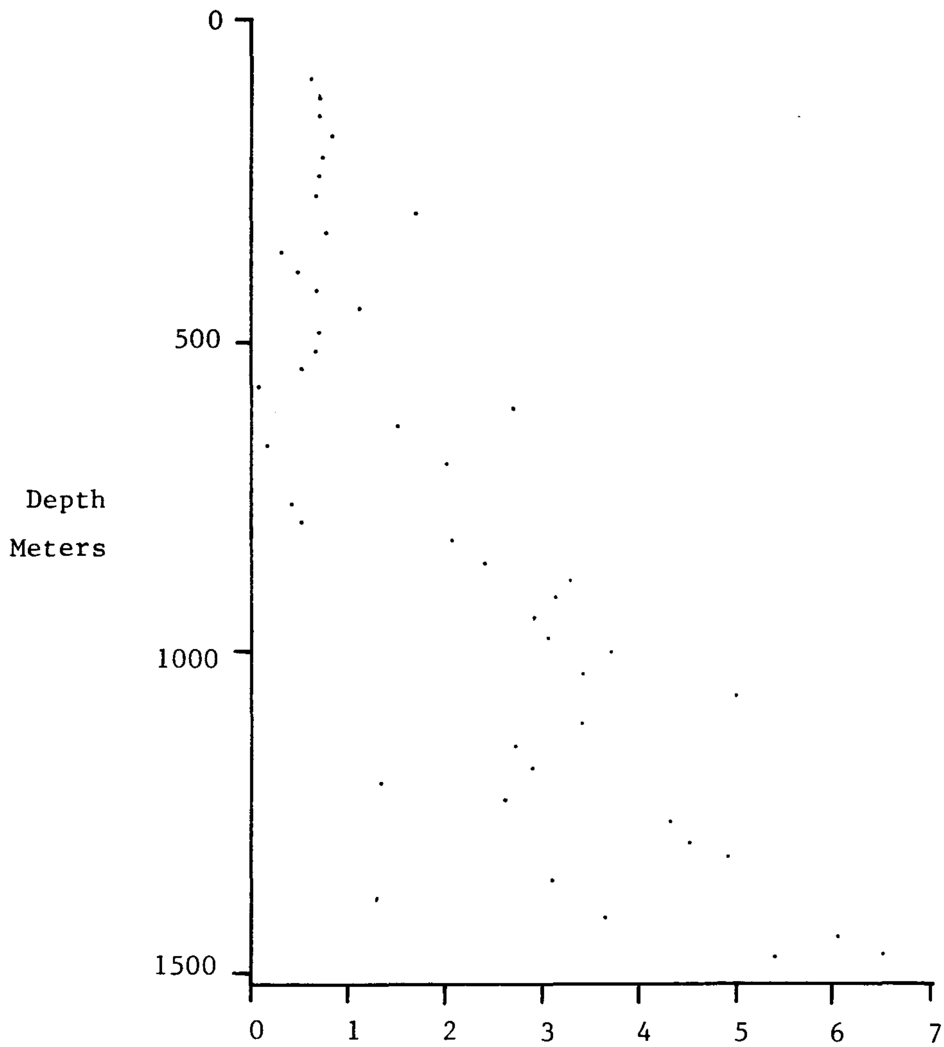

GRAPH 5: Horizontal Strain Rate X $10^{-11}$/sec




GRAPH 6: Vertical Strain Rate X $10^{-11}$/sec

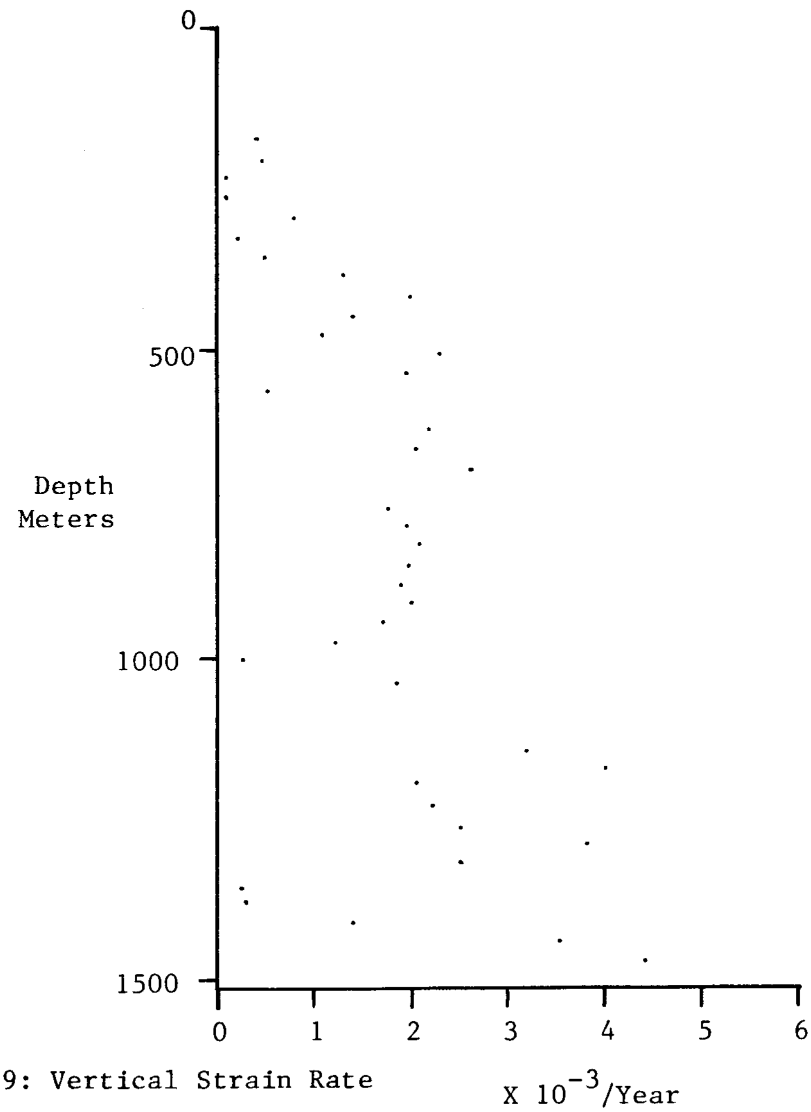



GRAPH 7: Velocity at Byrd Station   Meters/Year



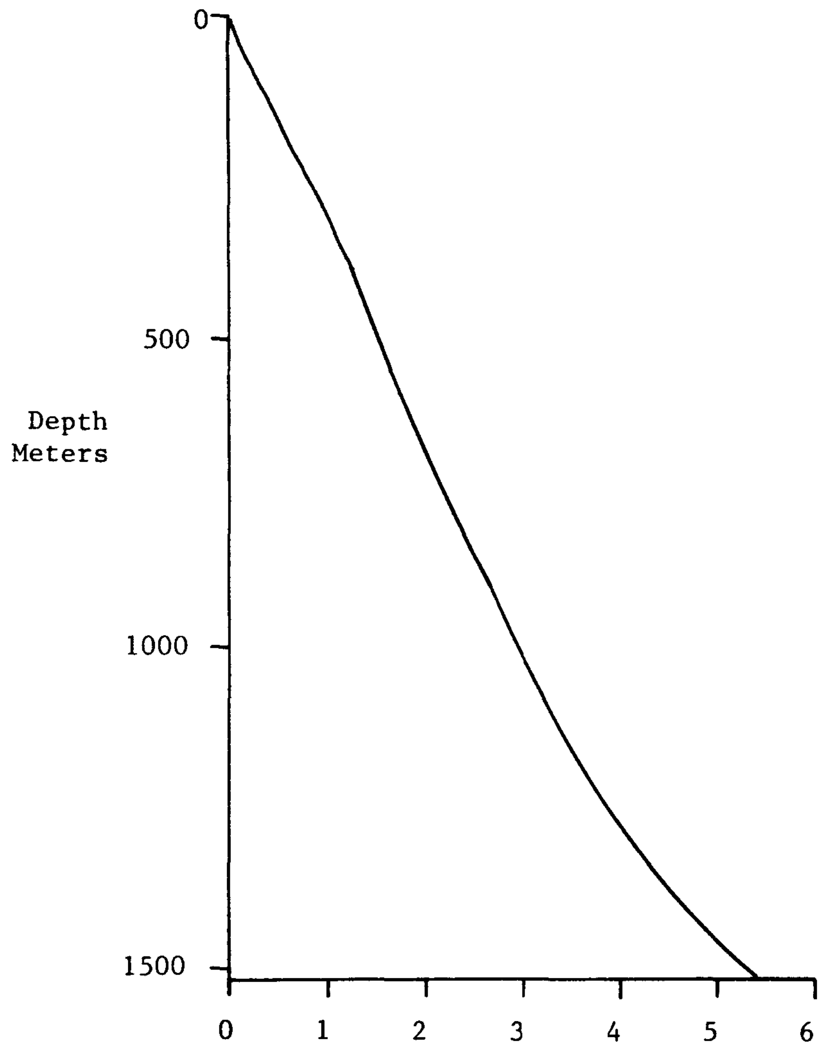

GRAPH 8: Vertical Strain Rate X-Component

X $10^{-3}$/Year

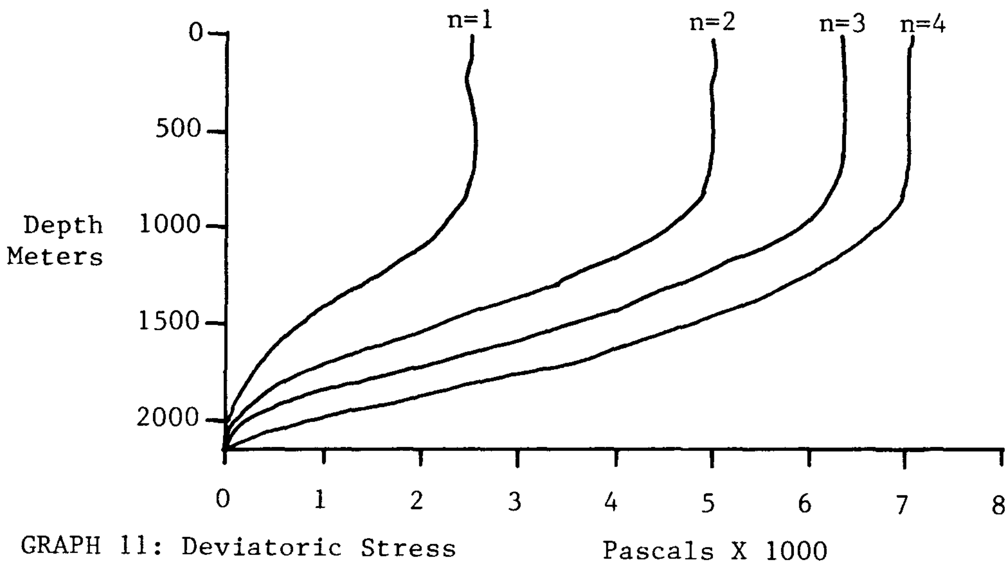



GRAPH 9: Vertical Strain Rate

Y-Component




GRAPH 10: Predicted Vertical Strain Rate X $10^{-3}$/Year




GRAPH 11: Deviatoric Stress
for values of n=1,2,3,4

REFERENCES:


Garfield, D.E., and Ueda, H.T., 1975, Resurvey of Byrd Station, Antarctica, drill hole. U.S. Cold Regions Research and Engineering Laboratory, Special Report 243, 11p.

Hooke, Roger LeB, 1981, Flow law for polycrystalline ice in glaciers: Comparison of theoretical predictions, laboratory data, and field measurements. Reviews of Geophysics and Space Physics, v.19, No.4, p.664-672

Kostecka, J.M., 1985, A numerical ice flow model applied to mass balance calculations in western Greenland, M.Sc. Thesis, The Ohio State Univ.

Lliboutry, L. and Duval, P.,1985,Various isotropic and anisotropic ices found in glaciers and polar ice caps and their corresponding rheologies. Annales Geophysicae, v.3, No. 2, p.207-224.

Nye, J.F. ,1959, The motion of ice sheets and glaciers. Journal of Glaciology, v.3, No.26, p.493-507.

Paterson, W.S B., 1981, The Physics of Glaciers, 2nd edition, Pergamon Press Ltd., Oxford, 380 p.

Paterson, W.S.B., 1983, Deformation within polar ice sheets: an analysis of Byrd Station and Camp Century borehole tilting measurements. Cold Regions Sci. and Tech. No.8, p. 165-179.

Van der Veen, C.J.,1985, Some thoughts about the stress distribution in ice sheets and ice shelves, Institute for Meteorology and Oceanography, State Univ. Utrecht, report R 85-4, 37 p.

Weertman, J.,1973, Creep of ice. In E. Whalley, S.J. Jones, and L.W. Gold (eds.), Physics and Chemistry of Ice. Roy. Soc. of Canada, Ottawa, Canada. p.320-337

Whillans, I.M., 1979, Ice flow along the Byrd Station strain network Antarctica, Journal of Glaciology, vol.24, No.90, p.15-28

Weast, Robert C. (editor), 1964, Handbook of Mathematical Tables, The Chemical Rubber Co., Cleveland, Ohio, 680 p.